

# Light-like Big Bang singularities in string and matrix theories

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## ABSTRACT

Important open questions in cosmology require a better understanding of the Big Bang singularity. In string and matrix theories, light-like analogues of cosmological singularities (singular plane wave backgrounds) turn out to be particularly tractable. We give a status report on the current understanding of such light-like Big Bang models, presenting both solved and open problems.

# 1 Introduction

The last two decades have witnessed enormous progress in our understanding of the composition and evolution of the universe. One of the remaining challenges is to understand how the very early universe reached a nearly homogeneous, nearly flat state with a specific spectrum of density perturbations consistent with present observations.

The most popular explanation is that the very early universe underwent a period of inflation [1]. If one assumes that inflation started and lasted long enough, it is able to explain the flatness and homogeneity of the universe. It also solves the monopole problem. The greatest success of inflation is that “simple” (single-field, slow-roll) inflationary models predict nearly scale-invariant, nearly Gaussian adiabatic density perturbations [2]. These are the seeds of large scale structure and are visible as temperature anisotropies in the cosmic microwave background. One may wonder, though, how the universe emerged in a state that allowed inflation to start. In other words, how was a suitably fine-tuned initial state selected? In particular, in general relativity, inflationary solutions are past geodesically incomplete (under a certain assumption that excludes a contracting phase in the past) [3]. The question should then be asked whether singularity resolution in a more fundamental theory puts constraints on which inflationary models are allowed.

Alternatives to inflation include the cyclic universe [4]. The ekpyrotic mechanism (ultra-slow contraction) generates a spectrum of perturbations very similar to that of inflation, but in a contracting universe [5]. In general relativity, the transition from a contracting to an expanding (spatially flat) universe requires going through a singularity [6]. At present, it is unclear whether such a transition is possible and whether perturbations would go through essentially unchanged. The answer will have to come from a theory beyond general relativity.

As we have argued, for the inflationary universe, and even more for alternative models, it is important to try and understand the big bang singularity. The work described in this review is motivated by several fundamental questions. Can we describe the big bang itself? How do space and time emerge from the big bang? Is it consistent to have a contracting universe before the big bang? Does the universe have a natural initial state, and if so, does it lead to inflation? String theory provides a short-distance modification of Einstein gravity, which is hoped to resolve space-time singularities. Existing formulations of string theory depend sensitively on the class of space-times one works with, and we will have to distinguish several classes of models.

Perturbative string theory requires the background space-time to be specified from the onset. This background space-time has to satisfy supergravity equations of motion with an infinite number of corrections expanded in powers of  $\alpha'$ , the inverse of the string tension. In the high curvature regime, which necessarily accompanies singularities, all these  $\alpha'$ -corrections generically become equally important, and the background equations of motion generically become completely intractable. One exception is provided by orbifolds, obtained from manifolds by discrete identifications [7]. Orbifolds contain new sectors of closed strings, namely “twisted” closed strings, which on the covering space connect a point and its image under a discrete identification. The rules of perturbative string theory on orbifolds are

inherited from those on the covering space. If the discrete identifications have fixed points, orbifolds can be singular. It is known that static orbifold singularities are resolved in perturbative string theory, precisely thanks to the inclusion of twisted closed strings becoming light near the singular point. The hope 10 years ago was that time-dependent orbifolds (see [8, 9, 10, 11] for reviews) would lead to simple examples of cosmological singularities resolved within perturbative string theory. It turned out, however, that at least the simplest models were plagued by divergences related to large backreaction and invalidating string perturbation theory [12, 13, 14, 15].<sup>1</sup> It is worth noting that, in perturbative string theory, lightlike orbifold singularities are equally problematic as spacelike ones.

One can therefore turn to non-perturbative formulations of quantum gravity, such as the AdS/CFT-correspondence [18] or matrix [19] or matrix string [20, 21, 22] theories. It should be said that those frameworks do not directly deal with the usual singularities discussed in cosmology. The AdS/CFT-correspondence requires the space-time to exhibit an AdS structure, at least asymptotically, while matrix and matrix string theories rely on the presence of a light-like isometry. None of these properties apply to the singularities naturally emerging in the context of classical cosmology: Friedmann, Kasner and Belinsky-Khalatnikov-Lifshitz space-times. Nevertheless, any progress on resolving light-like or space-like singularities, even in a not directly realistic model, would be very welcome, as it would point to mechanisms by which quantum gravity can in principle resolve cosmological singularities.

One clearly needs a compromise between the classes of space-times tractable within contemporary string theory and those relevant in cosmology. A number of directions of research have been proposed in this vein, and in this review we shall concentrate on one of them: analysis of light-like analogues of the usual singular cosmologies. For other models, we refer to the reviews [8, 9, 10, 11, 23, 24]. Within the class of light-like singularities, we will focus on studies in perturbative string theory and matrix (string) theory. For studies of light-like singularities using the AdS/CFT correspondence, see the review [25].

The prototype metric of the kind we are going to consider can be written as

$$ds^2 = -2dx^+dx^- + \mu_{ij}(x^+)dx^i dx^j. \quad (1)$$

When  $\mu$  is diagonal and depends on  $x^+$  through power-law functions, this looks very much like a Kasner solution, except that the dependences are on the light-cone time  $x^+$  rather than on the usual time. When  $\mu$  is proportional to the unit matrix with a power-law dependence on  $x^+$ , the metric looks like a light-like version of flat Friedmann cosmologies. Furthermore, (1) actually arises when a Penrose limit is taken around a light-like geodesic hitting cosmological singularities [26].

The metric (1) is of the plane wave type, i.e., it describes a strong plane-fronted gravitational wave (which is a non-linear generalization of the familiar linearized gravitational waves in Minkowski space-time). Such plane waves are most conveniently analyzed in the

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<sup>1</sup>See, however, [8] for a suggested resummation of divergences by working in the eikonal approximation, [16] for a proposed resolution of an orbifold singularity in terms of orientifolds, and [17] for a closely related model in which the singularity is replaced by a phase with a condensed winding tachyon within perturbative string theory.

so-called Brinkmann coordinates, with their metric given by

$$ds^2 = -2dx^+ dx^- + K_{ij}(x^+)x^i x^j (dx^+)^2 + (dx^i)^2, \quad (2)$$

where  $K_{ij}(x^+)$  represent the profiles of different polarization components of the wave. In pure gravity,  $K_{ij}(x^+)$  is constrained by  $K_{ii} = 0$ , giving the same number of polarizations as in linearized theory (a traceless symmetric tensor in  $D - 2$  dimensions, with  $D$  being the number of dimensions of space-time). If a dilaton is present,  $K_{ii}$  does not vanish and is related to the dilaton, which gives an additional independent polarization component. By a  $u$ -dependent rescaling of  $x^i$ , (2) can be brought to the so-called Rosen form, in which the metric only depends on  $u$ , making the planar nature of the wave front manifest. The Rosen form is implied in (1). (It is prone to coordinate singularities, however, and often avoided.)

Plane wave space-times possess a number of remarkable properties in the context of quantum gravity theories. In perturbative string theory, they are known to satisfy background consistency conditions to all orders in  $\alpha'$  [27]. In other words, if supergravity equations of motion (zeroth order in  $\alpha'$ ) are satisfied, all the higher order corrections will vanish automatically. This property comes from a special structure of the Riemann tensor in the plane wave background, and it allows perturbative string analysis of even highly curved plane waves. At the same time, the worldsheet theory of strings in plane wave backgrounds turns out to be especially simple (and linear, when the light cone gauge is imposed). Similarly, the light-like isometry needed for formulating matrix models is also present in plane wave geometries, so a matrix theory description of quantum effects in these space-times can be given.

It is most natural to start by studying (1) with  $\mu_{ij}$  proportional to the unit matrix (i.e., an isotropic space). Such space-times can be thought of as light-like analogues of Friedmann cosmologies. By the equations of motion, such space-times cannot be empty: one needs to add matter to compensate for the curvature of the plane wave. A natural choice for this additional matter is the dilaton, a scalar always present in string theories and supergravities. If one starts with (1) and takes  $\mu_{ij}$  to be proportional to  $\delta_{ij}$  and depending on  $x^+$  as a power law, the corresponding Brinkmann form metric (2) can be written as

$$ds^2 = -2dx^+ dx^- - \frac{k}{x^{+2}}(x^i)^2(dx^+)^2 + (dx^i)^2. \quad (3)$$

In turn, the equations of motion determine the dilaton to be [28]

$$\phi = \phi_0 + cX^+ + \frac{kd}{2} \ln x^+ \quad (4)$$

(where  $d = D - 2$  is the number of  $i$ -indices). Thus, for negative  $k$ , the string coupling  $e^\phi$  blows up at the singularity ( $x^+ = 0$ ), invalidating string perturbation theory. One would then expect a perturbative approach to be of little use for addressing the question of singularity transition in that case. It can still be applied, however, if  $k > 0$ . (Singularities with positive  $k$  arise as Penrose limits of power-law Big Bang singularities, with the scale factor of the universe proportional to positive powers of times; in the same way,  $k < 0$  corresponds to Big Rip singularities.)

It is often thought that quantum-gravitational effects should naturally resolve singularities in some way. This would certainly be desirable, but is not so in our explicit examples. The singular plane wave (3) enters various quantum gravity constructions as the background. The singularity is then always there at  $x^+ = 0$ , at least asymptotically, even though locally the geometry may be altered (or even dissolved by non-geometrical states). Mathematically, the singularity appears as explicit singular time dependence in the Hamiltonians of string and matrix theories in the plane wave (3). One then has to understand how to deal with such singular time dependences.

Free string propagation on (3) was studied in [28]. In particular, it was suggested in that publication that the question of propagation across the  $1/(X^+)^2$  singularity in the metric can be addressed by employing analytic continuation in the complex  $X^+$ -plane. Subsequently, in [29], another principle was proposed, which we motivate next.<sup>2</sup>

In string and matrix theories, it is necessary to satisfy stringent consistency conditions in order to maintain finiteness and anomaly cancellation. In perturbative string theories, this question is very well studied, and it is known that the space-time background has to satisfy  $\alpha'$ -corrected supergravity equations of motion in order for the theory to be well-defined (as already mentioned, the  $\alpha'$  corrections are absent if one is working with plane waves, hence satisfying plain supergravity equations of motion is sufficient). For matrix theories, similar restrictions arise from considering  $\kappa$ -symmetry of the D-brane action [31], though the question does not appear to have been studied systematically. In any case, one would expect that the handling of the singularity should be subject to rigid constraints (given that even for smooth space-times one encounters rigid constraints).

There is one approach to handling plane wave singularities that automatically takes benefit of what is known about smooth space-times and applies it to the singular limit. Namely, one can consider (3) as a limit of smooth metrics of the type (2), do the relevant computations, and take the singular limit at the end. Then, for any resolved space-time (2) consistency of string theory is guaranteed if (2) satisfies the supergravity equations of motion (without any further conditions). It is then natural to assume that the singular limit will likewise be a consistent string theory, provided that this limit exists.

Even with these specifications, there are many ways to resolve (3). One class of resolutions appears to be very special however. The background (3) possesses a scaling symmetry and does not depend on any dimensionful parameters. It is natural to demand that this symmetry should be recovered when the resolution is removed. This will happen if the resolved plane wave profile does not depend on any dimensionful parameters other than the resolution parameter  $\epsilon$ . In this case, on dimensional grounds,

$$K_{ij}(x^+, \epsilon) = -\delta_{ij} \frac{1}{\epsilon^2} \Omega(x^+/\epsilon). \quad (5)$$

The limit (3) will be recovered if

$$\Omega(\eta) \rightarrow \frac{k}{\eta^2} + O\left(\frac{1}{\eta^b}\right) \quad (6)$$

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<sup>2</sup>Recent work on closely related models appeared in [30].

for large values of  $\eta$ , with some  $b > 2$ .

In this review, we shall first concentrate on the perturbative string analysis of space-times (3), and then discuss how this class of backgrounds can be treated in matrix and matrix string theories.

## 2 Perturbative strings in singular plane wave backgrounds

### 2.1 The light cone Hamiltonian and WKB solutions

String worldsheet fermions are free in plane wave backgrounds [32]. We shall therefore concentrate on the bosonic part of the string action, given by

$$I = -\frac{1}{4\pi\alpha'} \int dt \int_0^{2\pi} d\sigma \sqrt{-g} \left( g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \alpha' R^{(2)} \Phi \right). \quad (7)$$

We choose light-cone gauge  $X^+ = \alpha' p^+ \tau$  and gauge-fix the worldsheet metric,

$$\det(g_{ab}) = -1, \quad \partial_\sigma g_{\sigma\sigma} = 0. \quad (8)$$

After some algebra and in units  $\alpha' = 1$ , one obtains the light-cone Hamiltonian as a sum of time-dependent harmonic oscillators:

$$H = \sum_{n=0}^{\infty} \sum_{i=1}^d H_{ni}, \quad (9)$$

$$H_{0i} = \frac{(P_{0i})^2}{2} + \frac{1}{\epsilon^2} \Omega(t/\epsilon) \frac{(X_0^i)^2}{2}, \quad (10)$$

$$H_{ni} = \frac{(P_{ni})^2 + (\tilde{P}_{ni})^2}{2} + \left( n^2 + \frac{1}{\epsilon^2} \Omega(t/\epsilon) \right) \frac{(X_n^i)^2 + (\tilde{X}_n^i)^2}{2}, \quad (11)$$

where  $X_n$  are Fourier transforms in the  $\sigma$ -coordinate:

$$X^i(t, \sigma) = X_0^i(t) + \sqrt{2} \sum_{n>0} \left( \cos(n\sigma) X_n^i(t) + \sin(n\sigma) \tilde{X}_n^i(t) \right). \quad (12)$$

The Hamiltonian (9) is quadratic and the solution to the corresponding Schrödinger equation can be found using WKB techniques, which are exact for quadratic Hamiltonians. The solution can be written as

$$\phi_n^i(t; X) = \mathcal{A}_n(t_1, t) \exp \left( i S_{cl;n} [X_{1,n}^i, t_1 | X_n^i, t] \right), \quad (13)$$

where  $S_{cl;n} [X_{1,n}^i, t_1 | X_n^i, t]$  is the “classical action” evaluated for the path going from  $X_{1,n}^i$  at the time  $t_1$  to  $X_n^i$  at the time  $t$ ,

$$S_{cl} [X_{1,n}^i, t_1 | X_n^i, t] = \int_{t_1}^t dt' \left( \frac{(\dot{X}_n^i)^2}{2} - \left( n^2 + \frac{1}{\epsilon^2} \Omega \left( \frac{t'}{\epsilon} \right) \right) \frac{(X_n^i)^2}{2} \right). \quad (14)$$

If  $\mathcal{A}_n(t_1, t)$  satisfies

$$-2\frac{\partial}{\partial t}\mathcal{A}_n(t_1, t) = \mathcal{A}_n(t_1, t)\frac{\partial^2}{\partial(X_n^i)^2}S_{cl}[X_{1,n}^i, t_1|X_n^i, t], \quad (15)$$

then (13) satisfies the original Schrödinger equation exactly.

Up to normalization, a basis of such solutions, labelled by the initial condition  $X_n^i(t_1) = X_{1,n}^i$ , is given by [33]

$$\phi(t; X_n^i) \sim \prod_{ni} \frac{1}{\sqrt{\mathcal{C}(t_1, t)}} \exp\left(-\frac{i}{2\mathcal{C}} \sum_{i=1}^d [(X_{1,n}^i)^2 \partial_{t_1} \mathcal{C} - (X_n^i)^2 \partial_{t_2} \mathcal{C} + 2X_{1,n}^i X_n^i]\right), \quad (16)$$

where  $\mathcal{C}(t_1, t_2)$  (suppressing the index  $n$ ) is a solution to the “classical equation of motion” for the time-dependent harmonic oscillator Hamiltonian (11):

$$\partial_{t_2}^2 \mathcal{C}(t_1, t_2) + \left(n^2 + \frac{1}{\epsilon^2} \Omega(t_2/\epsilon)\right) \mathcal{C}(t_1, t_2) = 0, \quad (17)$$

with initial conditions specified as

$$\mathcal{C}(t_1, t_2)|_{t_1=t_2} = 0, \quad \partial_{t_2} \mathcal{C}(t_1, t_2)|_{t_1=t_2} = 1. \quad (18)$$

We shall refer to  $\mathcal{C}(t_1, t_2)$  as “compression factor”. To derive the singular limit of the wavefunction (16) it is sufficient to study the singular limit of (17-18).

## 2.2 The singular limit for the center-of-mass motion

For the  $n = 0$  mode, we obtain as the “classical equation of motion”

$$\ddot{X} + \frac{1}{\epsilon^2} \Omega(t/\epsilon) X = 0. \quad (19)$$

We need to study the  $\epsilon \rightarrow 0$  limit of the solution that obeys the initial conditions

$$X(t_1) = 0, \quad \dot{X}(t_1) = 1, \quad t_1 < 0. \quad (20)$$

The singular limit of solutions to this equation has been analyzed in [33]. Performing a scale transformation  $Y(\eta) = X(\eta\epsilon)$ , with  $\eta = t/\epsilon$ , removes the  $\epsilon$ -dependence from the equation, leaving

$$\frac{\partial^2}{\partial \eta^2} Y + \Omega(\eta) Y = 0. \quad (21)$$

This scale transformation is possible because our initial singular metric was scale-invariant and we have resolved it as in (5) without introducing any dimensionful parameters besides  $\epsilon$ . The existence of a singular limit is then translated [33] into constraints on the asymptotic behavior of solutions to (21). These “boundary conditions at infinity” are strongly

reminiscent of a Sturm-Liouville problem, and it is natural that a discrete spectrum for the overall normalization of  $\Omega$  is singled out by imposing the existence of a singular limit.

For the specific asymptotics of our resolved profile (6), it can be shown [33] that, in the infinite past and infinite future, the solutions approach a linear combination of two powers (denoted below  $a$  and  $1-a$ , with  $a$  being a function of  $k$ , cf. (5-6)). This power law behavior simply corresponds to the regime when the second term on the right hand side of (6) can be neglected compared to the first. It is then convenient to form two bases of solutions, one asymptotically approaching the two powers (dominant and subdominant) at  $\eta \rightarrow -\infty$ ,

$$Y_{1-}(\eta) = |\eta|^{a-} + o(|\eta|^{a-}), \quad Y_{2-}(\eta) = |\eta|^{1-a-} + o(|\eta|^{1-a-}), \quad (22)$$

and another behaving similarly at  $\eta \rightarrow +\infty$

$$Y_{1+}(\eta) = |\eta|^{a+} + o(|\eta|^{a+}), \quad Y_{2+}(\eta) = |\eta|^{1-a+} + o(|\eta|^{1-a+}), \quad (23)$$

where  $a_{\pm}$  is given by

$$a_{\pm} = \frac{1}{2} + \sqrt{\frac{1}{4} - k_{\pm}}. \quad (24)$$

(We are temporarily assuming that  $k$  can take two different values  $k_{\pm}$  for the positive and negative time asymptotics, a possibility that will be discarded shortly.) The two bases are related by a linear transformation:

$$\begin{bmatrix} Y_{1-}(\eta) \\ Y_{2-}(\eta) \end{bmatrix} = Q \begin{bmatrix} Y_{1+}(\eta) \\ Y_{2+}(\eta) \end{bmatrix}, \quad (25)$$

where  $Q$  is a  $2 \times 2$  matrix whose determinant is constrained by Wronskian conservation as

$$W[Y_{1-}, Y_{2-}] = W[Y_{1+}, Y_{2+}] \det Q. \quad (26)$$

The singular limit has been rigorously considered in [33], but the results can be understood heuristically from the following argument [29]. Imagine one is trying to construct a solution  $\tilde{Y}$  to (21) satisfying some ( $\epsilon$ -independent) initial conditions at  $\eta_1 = t_1/\epsilon < 0$ . This solution can be expressed in terms of  $Y_{1-}$  and  $Y_{2-}$  (a complete basis) as

$$\tilde{Y} = C_1 Y_{1-} + C_2 Y_{2-}. \quad (27)$$

Since the initial conditions are specified at  $\eta_1 = t_1/\epsilon$ , the asymptotic expansions (22) are valid. There needs to be a non-trivial contribution from both  $Y_{1-}$  and  $Y_{2-}$  in the above formula in order to satisfy general initial conditions. Hence, the two terms on the right hand side should be of order 1. Therefore, we should have

$$C_1 = O(\epsilon^{a-}), \quad C_2 = O(\epsilon^{1-a-}). \quad (28)$$

If we now apply (25) and (23) to evaluate  $\tilde{Y}$  at a large positive  $\eta = t_2/\epsilon$ , the powers of  $\epsilon$  in  $C_1$  and  $C_2$  will combine with the powers of  $\epsilon$  originating from  $Y_{1+}$  and  $Y_{2+}$  and yield

$$\begin{aligned} \tilde{Y}(t_2/\epsilon) = & Q_{11} t_2^{a+} O(\epsilon^{a--a+}) + Q_{12} t_2^{1-a+} O(\epsilon^{a-+a+-1}) \\ & + Q_{21} t_2^{a+} O(\epsilon^{1-a--a+}) + Q_{22} t_2^{1-a+} O(\epsilon^{a+-a-}). \end{aligned} \quad (29)$$



Since  $a_+$  and  $a_-$  are greater than  $1/2$ , this expression can only have an  $\epsilon \rightarrow 0$  limit if  $a_+ = a_-$  (i.e.,  $k_+ = k_-$ ) and  $Q_{21} = 0$ . The latter condition implies that the overall normalization of the plane wave profile  $\Omega(\eta)$  will generically lie in a discrete spectrum, dependent on the specific way the singularity is resolved, i.e., the detailed shape of  $\Omega(\eta)$ . A particular exactly solvable example for this discrete spectrum (there called “light-like reflector plane”) has been given in [34]. With  $Q_{21} = 0$  and  $\det Q = -1$ , the matrix  $Q$  can be written as

$$Q = \begin{bmatrix} q & \tilde{q} \\ 0 & -1/q \end{bmatrix}, \quad (30)$$

with  $q$  being a real nonzero number ( $\tilde{q}$  does not affect the singular limit). For flat space-time we have  $q = 1$  and for the “light-like reflector plane” of [34] we have  $q = -1$ . In the singular limit, a basis of solutions is given by

$$\begin{aligned} Y_1(t) &= (-t)^a, & Y_2(t) &= (-t)^{1-a}, & t < 0, \\ Y_1(t) &= q t^a, & Y_2(t) &= -\frac{1}{q} t^{1-a}, & t > 0. \end{aligned} \quad (31)$$

### 2.3 The singular limit for excited string modes

Following our general discussion of free strings in plane wave backgrounds, the evolution of excited string modes is described by time-dependent harmonic oscillator equations

$$\frac{\partial^2}{\partial t^2} X(t) + \left( n^2 + \frac{1}{\epsilon^2} \Omega(t/\epsilon) \right) X(t) = 0. \quad (32)$$

Solutions for the wavefunctions of the excited string modes can be expressed in terms of a particular solution to this equation  $\mathcal{C}(t_1, t_2)$  defined by (17-18). Hence, to analyze the singular ( $\epsilon \rightarrow 0$ ) limit of the excited modes dynamics, it should suffice to analyze the singular limit of  $\mathcal{C}(t_1, t_2)$ . Because  $n^2$  is finite, it is natural to expect that it does not affect the existence of the singular limit (governed by the singularity emerging from  $\Omega(t/\epsilon)$ ). It can be proved that it is indeed the case for positive  $k$  [29].

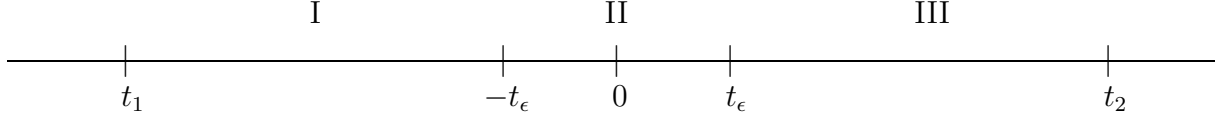
The general strategy here is to analyze (32) separately in the near-singular region ( $t$  close to 0) and the region where the  $\epsilon \rightarrow 0$  limit is regular. Away from  $t = 0$ , up to corrections vanishing as  $\epsilon$  is taken to 0, (32) can be approximated by

$$\ddot{X}(t) + \left( n^2 + \frac{k}{t^2} \right) X(t) = 0, \quad (33)$$

which is related to Bessel’s equation. Around  $t = 0$ , one should expect that  $n^2$  can be neglected, which leaves the equation for the zero-mode (already analyzed in the previous section):

$$\ddot{X}(t) + \frac{1}{\epsilon^2} \Omega(t/\epsilon) X(t) = 0. \quad (34)$$

More specifically, the separation into near-singular and regular regions should be organized as follows:



We use  $t_\epsilon$  to indicate a time that will approach zero in the singular limit as

$$t_\epsilon = \epsilon^{1-c} \tilde{t}^c, \quad (35)$$

with  $\tilde{t}$  staying finite in relation to the “moments of observation”  $t_1$  and  $t_2$ . The number  $c$  (between 0 and 1) should be chosen later as needed for our proof.

One then can show that (for positive  $k$ ) there is indeed a choice of  $c$  that makes deviations from the approximate equations (33-34) small in the appropriate regions. One can then construct the  $\epsilon \rightarrow 0$  limit of solutions to (32) by taking approximate solutions satisfying (33) and (34), splicing them together and taking the  $\epsilon \rightarrow 0$  limit in the end. Thus, one obtains exact expressions for the  $\epsilon \rightarrow 0$  limit of solutions to (32), even though analytic solutions to (32) at finite  $\epsilon$  cannot be given.

We refer the reader to the original article [29] for detailed proofs, and here simply state the result: For  $k > 0$ , the singular limit of the excited mode evolution exists whenever it exists for the center-of-mass motion, and it is described by the following matching conditions:

$$\begin{aligned} Y_1(t) &= \sqrt{-t} J_{a-1/2}(-nt), & Y_2(t) &= \sqrt{-t} J_{1/2-a}(-nt), & t < 0, \\ Y_1(t) &= q\sqrt{t} J_{a-1/2}(nt), & Y_2(t) &= -\frac{\sqrt{t}}{q} J_{1/2-a}(nt), & t > 0, \end{aligned} \quad (36)$$

where  $J_\nu$  are Bessel functions.

Note that there is a slight subtlety in the sense that convergence to the  $\epsilon \rightarrow 0$  limit is not uniform with respect to  $n$  (we have kept  $n$  fixed in our considerations). However, this does not affect the result as long as one is only interested in the limiting expressions at  $\epsilon = 0$ . More discussion is given in the original publication [29].

## 2.4 The singular limit for the entire string

As we have seen in the previous section, for  $k > 0$ , consistent propagation of the string center-of-mass across the singularity guarantees that all excited string modes also propagate in a consistent fashion. This is not sufficient, however, to define a consistent evolution for the whole string, since even small excitations of higher string modes can sum up to yield an infinite total energy [27]. As we shall see below, the condition of finite total string energy (after the singularity crossing) turns out to be very restrictive.

The total string excitation energy can be conveniently expressed in terms of the Bogoliubov coefficients for the higher string modes which can be extracted from (36) as

$$\alpha_n = -\frac{1+q^2}{2q \sin(\alpha\pi)}, \quad (37)$$

$$\beta_n = i \frac{\exp(-i\pi\alpha) + q^2 \exp(i\pi\alpha)}{2q \sin(\alpha\pi)}, \quad (38)$$

and they turn out to be independent of  $n$ . Here,  $\alpha = \sqrt{1-4k}/2$ . The total mass of the string after crossing the singularity is given by [27]

$$M = \sum_n n |\beta_n|^2. \quad (39)$$

Since the  $\beta_n$  are  $n$ -independent,  $M$  can only be finite if  $\beta_n = 0$  for all  $n$ . For  $k > 0$ , this cannot be achieved, since  $0 < \alpha < 1/2$  and  $q$  is real. (For  $k = 0$ , which is the case of the “lightlike reflector plane” of [34], all  $\beta_n$  will vanish if  $q^2 = 1$ , which is satisfied automatically for any reflection-symmetric  $\Omega$ .) The implication is then that, if the singularity is resolved without introducing any new dimensionful scales, the singularity transition cannot be defined. (The alternative is having dimensionful parameters buried strictly at the singular locus, and an explanation of the possible physical origin of such parameters would be in order.)

As mentioned before, in the case  $k < 0$  the string coupling blows up near the singularity, so that string perturbation theory is certainly not valid. Our considerations can be seen as a motivation to study these backgrounds in the context of non-perturbative formulations of string theory, to which we now turn.

### 3 Matrix big bang models

#### 3.1 Time-dependent matrix and matrix string theories

Matrix theory [19] is a non-perturbative formulation of M-theory in 11-dimensional Minkowski space-time. One way to derive it is by Discrete Light-Cone Quantization (DLCQ) [35]. A sector with  $N$  units of lightcone momentum is described by a quantum mechanics of  $N \times N$  matrices, namely the dimensional reduction of (9+1)-dimensional  $SU(N)$  super-Yang-Mills theory to 0+1 dimensions. In matrix theory, (lightcone) time is built in, but space is an emergent concept, arising from a “moduli space” of flat directions. For this “moduli space”, and therefore space-time, to emerge, supersymmetry plays an essential role: without supersymmetry, quantum corrections would lift the flat directions. Compactifying M-theory on a circle leads to type IIA string theory. Matrix string theory [20, 21, 22] is a non-perturbative formulation of type IIA string theory in 10d Minkowski space-time. The aspects of matrix and matrix string theory needed for our purposes are reviewed in detail in [10].

The construction of matrix theories essentially relies on compactifying a light-like direction (which is a pre-requisite for discrete light-cone quantization). Due to discreteness, positivity and conservation of the light-cone momenta in such a compactified space-time, Hilbert spaces of quantum theories living in this space-time split into independent sectors (labelled by the value of the light-cone momentum), each of which corresponds to a quantum system with a *finite* number of degrees of freedom. For matrix theories, this quantum system is given explicitly by finite  $N$  matrix Lagrangians. This is a remarkable simplification (which automatically renders the theory UV-finite, since there are no divergences in quantum mechanics). The drawback is equally grave, however, as reconstructing quantities

in an infinite space-time from the compactified version is highly non-trivial (it is believed to be possible, however, since for a sufficiently large compactification radius, an arbitrarily large laboratory can fit into the space-time with a light-like compactification).

Compactifying a light-like direction is only possible in space-time backgrounds with a light-like isometry. This obviously excludes the case of ordinary cosmological space-times, as well as almost all familiar non-trivial solutions to the classical gravitational equations of motion. Strong plane waves of the type (1, 2) are an exception though, which highlights once again their special status in quantum gravity and makes them an interesting laboratory for applying matrix theory methods to study time dependence and strong gravitational effects.

One must keep in mind that reconstructing quantities in a decompactified space-time from their DLCQ analogues (explicitly given by the finite  $N$  matrix theories) may become more subtle in a time-dependent setting.<sup>3</sup> The point is that, at best, DLCQ can provide information on quantities in a decompactified space that are measurable in a finite-size laboratory. This is not likely to pose a problem for the case of mild time dependences. But for the opposite extreme, namely space-times including singularities (which we shall comment on in the next section), linear dimensions of physical systems in an infinite space may blow up indefinitely. In that case, finite box dynamics (and hence DLCQ) will not be adequate to describe the evolution in a decompactified space. Whether or not these subtleties do arise has to be decided on the basis of careful dynamical considerations, which have not been carried out yet.

We shall now give a summary of a few different versions of matrix and matrix string theories in plane wave backgrounds. For the original time-dependent Matrix Big Bang matrix string theory of [36], the 10-dimensional geometry is asymptotic to the linear dilaton configuration:<sup>4</sup>

$$\begin{aligned} ds_{st}^2 &= -2dy^+ dy^- + (dy^i)^2, \\ \phi &= -Qy^+. \end{aligned} \tag{40}$$

To construct the matrix string theory for the background (40), one first lifts the background (40) to 11 dimensions via the usual conjecture of type IIA/M-theory correspondence. The resulting 11-dimensional space-time is

$$ds^2 = e^{2Qy^+/3} (-2dy^+ dy^- + (dy^i)^2) + e^{-4Qy^+/3} (dy)^2, \tag{41}$$

where  $y$  is a coordinate along the M-theory circle. This is followed by the DLCQ compactification of the light-like  $v$ -coordinate, interpreted as the M-theory circle of an “auxiliary” type IIA string theory. A T-duality [48] then relates the resulting theory of D0-branes on a compact dimension, i.e., a BFSS-like matrix theory with a compactified dimension, to a more manageable theory of wrapped D1-branes. This procedure has been carried out (in a slightly different but equivalent way) in [36] and has been reviewed in [10] and [47]. The

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<sup>3</sup>We thank David Kutasov for drawing our attention to this issue.

<sup>4</sup>The original set-up of [36] has been later extended in various directions [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]; in particular, a systematic generalization of the analysis to more general singular homogeneous plane-wave space-time backgrounds has appeared in [47].

resulting matrix string action is

$$S = \frac{1}{2\pi\ell_s^2} \int \text{tr} \left( \frac{1}{2} (D_\mu X^i)^2 + \theta^T \bar{D} \theta + \frac{1}{4g_{YM}^2} F_{\mu\nu}^2 - g_{YM}^2 [X^i, X^j]^2 + g_{YM} \theta^T \gamma_i [X^i, \theta] \right), \quad (42)$$

with the Yang-Mills coupling  $g_{YM}$  related to the worldsheet values of the dilaton:

$$g_{YM} = \frac{e^{-\phi(y^+(\tau))}}{2\pi l_s g_s} = \frac{e^{Q\tau}}{2\pi l_s g_s}. \quad (43)$$

A generalization of this set-up has been proposed [47]. One can start with a 10-dimensional power-law plane wave:

$$\begin{aligned} ds_{st}^2 &= -2dy^+ dy^- + g_{ij}(y^+) dy^i dy^j \equiv -2dy^+ dy^- + \sum_i (y^+)^{2m_i} (dy^i)^2 \\ &= -2dz^+ dz^- + \sum_a \frac{m_a(m_a - 1)}{(z^+)^2} (z^a)^2 (dz^+)^2 + \sum_a (dz^a)^2, \\ e^{2\phi} &= (y^+)^{3b/(b+1)} = (z^+)^{3b/(b+1)}. \end{aligned} \quad (44)$$

Here, the first and the second line represent the Rosen and the Brinkmann form of the same plane wave, respectively. In order for the supergravity equations of motion to be satisfied, one needs to impose [47]

$$\sum_i m_i(m_i - 1) = -\frac{3b}{b+1}. \quad (45)$$

The original background of [36] can be seen as a  $b \rightarrow -1$  limit of the above space-time [47]. The 11-dimensional space-time corresponding to (44) is

$$\begin{aligned} ds_{11}^2 &= -2dudv + \sum_i u^{2n_i} (dy^i)^2 + u^{2b} (dy)^2 \\ &= -2dudw + \sum_a \frac{n_a(n_a - 1)}{u^2} (x^a)^2 (du)^2 + \frac{b(b-1)}{u^2} x^2 (du)^2 + \sum_a (dx^a)^2 + (dx)^2, \end{aligned} \quad (46)$$

with  $n_i$  related to  $m_i$  by  $2m_i = (2n_i + b)/(b+1)$ . The usual formulation leads to a matrix string action, whose bosonic part is given, in the Rosen coordinates of (44), by

$$\begin{aligned} S_{RC} = \int d\tau d\sigma \text{Tr} \left( -\frac{1}{4} g_{YM}^{-2} \eta^{\alpha\gamma} \eta^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} - \frac{1}{2} \eta^{\alpha\beta} g_{ij}(\tau) D_\alpha X^i D_\beta X^j \right. \\ \left. + \frac{1}{4} g_{YM}^2 g_{ik}(\tau) g_{jl}(\tau) [X^i, X^j] [X^k, X^l] \right). \end{aligned} \quad (47)$$

with the transverse metric  $g_{ij}$  given by the first line of (44) and the Yang-Mills coupling by

$$g_{YM} = \frac{e^{-\phi(y^+(\tau))}}{2\pi l_s g_s} = \frac{\tau^{-3b/2(b+1)}}{2\pi l_s g_s}. \quad (48)$$

One can further transform this action to the Brinkmann coordinates of the original plane wave, given by the second line of (44), to obtain [47]:

$$S_{BC} = \int d\tau d\sigma \text{Tr} \left( -\frac{1}{4} g_{YM}^{-2} F_{\tau\sigma}^2 - \frac{1}{2} (D_\tau Z^a D_\tau Z^a - D_\sigma Z^a D_\sigma Z^a) \right. \\ \left. + \frac{1}{4} g_{YM}^2 [Z^a, Z^b][Z^a, Z^b] + \frac{1}{2} A_{ab}(\tau) Z^a Z^b \right), \quad (49)$$

where  $A_{ab} = \text{diag}\{m_a(m_a - 1)\}/\tau^2$ . The latter form of the action only differs from a SYM gauge theory with a time-dependent coupling by the term involving  $A_{ab}$ .

In [37], 11-dimensional (quantum-mechanical) matrix theories were introduced as simpler analogues of the matrix string theories of [36]. The relevant 11-dimensional (M-theory) background has the form

$$ds^2 = e^{2\alpha x^+} (-2dx^+ dx^- + (dx^i)^2) + e^{2\beta x^+} (dx^{11})^2, \quad (50)$$

or, in terms of the light-like geodesic affine parameter  $\tau = e^{2\alpha x^+}/2\alpha$ ,

$$ds^2 = -2d\tau dx^- + 2\alpha\tau (dx^i)^2 + (2\alpha\tau)^{\beta/\alpha} (dx^{11})^2. \quad (51)$$

This metric satisfies the 11-dimensional supergravity equations of motion if the constants  $\alpha$  and  $\beta$  are related as  $\beta = -2\alpha$ , or  $\beta = 4\alpha$ . The fact that these relations need to be imposed will not be relevant for what follows (it is essential, however, for the general consistency of the corresponding matrix theories). Since translations in  $x^-$  form an isometry of the above background, the usual DLCQ argument (proposed in [49] and adapted to the time-dependent case in [36]) can be applied. The result [37] is a matrix theory that can be expected to describe non-perturbative quantum gravity in space-times asymptotic to (51). The bosonic and fermionic parts of the matrix theory action, respectively, have the following form:

$$S_B = \int d\tau \text{Tr} \left\{ \frac{\alpha\tau}{R} (D_\tau X^i)^2 + \frac{(2\alpha\tau)^{\beta/\alpha}}{2R} (D_\tau X^{11})^2 - \frac{R}{4} (2\alpha\tau)^2 [X^i, X^j]^2 \right. \\ \left. - \frac{R}{2} (2\alpha\tau)^{1+\beta/\alpha} [X^i, X^{11}]^2 \right\}, \quad (52)$$

$$S_F = \int d\tau \{ i\theta^T D_\tau \theta - R\sqrt{2\alpha\tau} \theta^T \gamma_i [X^i, \theta] - R(2\alpha\tau)^{\beta/2\alpha} \theta^T \gamma_{11} [X^{11}, \theta] \}.$$

### 3.2 Singularity transition

The models we have just formulated are aimed to describe quantum-gravitational effects in singular plane waves (46). In Rosen coordinates (first line of (46)), these space-times can be seen as a light-like analogue of Friedmann or Kasner cosmologies. It is then interesting to investigate what the theory has to say about the question of singularity transition.

The general structure that emerges from our considerations is matrix (string) Hamiltonians with singular time-dependences. This can be explicitly seen in (49) as  $A_{ab}$  is singular

at  $\tau = 0$ . Likewise, the factors of  $\tau$  in front of the time derivatives in (52) will become inverted when the velocities are replaced by momenta, and the corresponding Hamiltonian will contain an isolated singularity in its time dependence at  $\tau = 0$ . These isolated singularities in the explicit time dependence of the matrix (string) Hamiltonians are directly inherited from the singularity in the background space-time. (Such singularities are also present in Hamiltonians (10-11) in the perturbative string theory setting of the previous section, if one looks at the  $\varepsilon \rightarrow 0$  singular limit directly without performing the singularity resolution.)

One might have thought that quantum gravity should resolve singularities in some way and give a dynamical prescription for singularity transitions. Unfortunately, it is not so in our present setting. The underlying space-time singularities appear as explicit singularities in time dependences of matrix (string) Hamiltonians, and the evolution is not well-defined without additional prescriptions.

Some general properties of Hamiltonians with singular time-dependences have been studied in [50, 34]. An important thing to understand is that there is a tremendous ambiguity associated with defining the singularity transition in this context. Indeed, if one only assumes that the Schrödinger equation for the Hamiltonian with a singular time dependence is satisfied away from singularity, and finds a way to define the singularity transition, one can immediately create another singularity transition prescription with the same properties. Namely, one can change the wave vectors by an arbitrary unitary transformation the moment they pass the singularity. The Schrödinger equation will still be satisfied away from the singularity, as it was before. One is, therefore, in a need of a physical (or at least heuristic) motivation for choosing a particular prescription for singularity transition.

We believe that a very natural class of singularity transitions arises from considering geometrical resolutions of the singular plane wave (46). Due to the functional arbitrariness of the plane wave profile, it is very easy to replace the singular profile in (46) by a resolved one in a way similar to (5-6). Maintaining the background space-time as a solution to the equations of motion is also essential from the standpoint of background consistency. Enforcing supergravity equations for the background appears to be related to the  $\kappa$ -symmetry of the D-brane action [31] necessary for the standard formulation of matrix theories. Unlike the case of perturbative strings, the action is non-linear, so the singular limit may be much more difficult to investigate, and it remains an important open problem for the future.

(Note the similarities of this setting to the AdS light-like cosmologies reviewed in [25]. There, one ends up with a gauge theory featuring singular light-like time dependences. The only difference in our case is that the singular dependences are on ordinary time. Similar non-linearities characteristic of gauge theories are present in both cases. In [25], a special class of plane waves is chosen for which the analysis simplifies due to conformal flatness. General p-brane-plane-wave solutions presented in [51] suggest a natural generalization of AdS light-like cosmologies featuring arbitrary plane wave profiles. In that setting, the problem of singularity transition is even more similar to what one encounters in Matrix Big Bang models.)

### 3.3 Near-classical late time dynamics

Even though, in the context of time-dependent matrix and matrix string theories, novel physics is expected to emerge in the high-curvature regions of space time, it is also important to understand how the near-classical space-time emerges away from the singularity when the curvature becomes small. The issue may be technically somewhat involved, since geometrical notions appear rather indirectly in the matrix formalism. Heuristic results appeared in [36, 52]; a systematic analysis has recently been carried out in [53], the results of which we now review.<sup>5</sup>

The problems of late time (near-classical) dynamics in matrix models can be understood by inspecting the action (52). The space-time backgrounds implicit in (52) are supposed to feature a light-like singularity at  $\tau = 0$  and become progressively more classical at large  $\tau$ . Yet, the explicit time dependences in (52) superficially become more steep, if anything, at large  $\tau$ . Additionally, there are no supersymmetries explicit in (52). Since supersymmetries are crucial for the free propagation of well-separated gravitons (and hence, a robust geometrical interpretation) in the flat space matrix theory, one should attempt to find an analogue of supersymmetry in (52) that would enforce a similar type of dynamics.

To address these important issues, we first note that the metric (51) describes a plane wave and, with the coordinate transformation

$$u = \tau, \quad z^i = \sqrt{2\alpha\tau}x^i, \quad z^{11} = (2\alpha\tau)^{\beta/2\alpha}x^{11}, \quad v = x^- + \frac{\alpha(z^i)^2 + \beta(z^{11})^2}{4\alpha\tau}, \quad (53)$$

it can be brought to the Brinkmann form

$$ds^2 = -2du dv - \frac{\alpha^2(z^i)^2 - (\beta^2 - 2\alpha\beta)(z^{11})^2}{(2\alpha u)^2} du^2 + (dz^i)^2 + (dz^{11})^2. \quad (54)$$

This parametrization forces the metric to manifestly approach Minkowski space-time for large values of the light-cone time, which strongly suggests that the large time dynamics of the corresponding matrix theory will likewise approach the flat space matrix theory, if treated in appropriate variables. (As we shall see below, the convergence towards this limit is somewhat subtle, but the naïve expectation will prove well-grounded.)

The matrix theory corresponding to (54) is given by

$$S_B = \int d\tau \text{Tr} \left\{ \frac{1}{2R} [(D_\tau Z^i)^2 + (D_\tau Z^{11})^2] - \frac{R}{4} [Z^i, Z^j]^2 + 2[Z^i, Z^{11}]^2 \right. \\ \left. - \frac{\alpha^2(Z^i)^2 - (\beta^2 - 2\alpha\beta)(Z^{11})^2}{(2\alpha\tau)^2} \right\}, \quad (55)$$

$$S_F = \int d\tau \{ i\theta^T D_\tau \theta - R\theta^T \gamma_i [Z^i, \theta] - R\theta^T \gamma_{11} [Z^{11}, \theta] \}.$$

The action (55) only differs from the flat space matrix theory by a term decaying as  $1/\tau^2$ , thus one may expect that the late time dynamics will be approximated by the flat

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<sup>5</sup>Another attempt to study late time (low background curvature) dynamics of the time-dependent matrix theories was undertaken in [54].



space matrix theory and admit the usual space-time interpretation. However, the decay is quite slow and one might be worried about whether it is sufficient to ensure convergence.

To illustrate these worries, one may look at the straightforward example of a harmonic oscillator whose frequency depends on time as  $1/t^2$ :

$$\ddot{x} + \frac{k}{t^2}x = 0. \quad (56)$$

The two independent solutions to this equation can be given as  $t^a$  and  $t^{1-a}$ , where  $a$  is a  $k$ -dependent number. These two solutions are obviously quite different from a free particle trajectory, even though the equation of motion approaches that of a free particle at late times. The reason for this discrepancy is the slow rate of decay of the second term in (56).

However, in a physical setting, one is only able to perform *finite time* experiments. That is, one has to specify the initial values  $x(t_0) = x_0$ ,  $\dot{x}(t_0) = v_0$  and examine the corresponding solution between  $t_0$  and  $t_0 + T$ . The solution is given by

$$x(t) = \frac{x_0(1-a) - v_0 t_0}{1-2a} \left(\frac{t}{t_0}\right)^a + \frac{v_0 t_0 - x_0 a}{1-2a} \left(\frac{t}{t_0}\right)^{1-a}. \quad (57)$$

One can then see that  $x(t_0 + T) = x_0 + v_0 T + O(T/t_0)$ , i.e., it is approximated by a free motion arbitrarily well if the experiment starts sufficiently late.

It may be legitimately expected that the *finite time* behavior of the full time-dependent matrix theory given by (55) will be approximated arbitrarily well by the flat space matrix theory at late times, just as in the above harmonic oscillator example. We shall now prove it by constructing an elementary bound on dynamical deviations due to a small time-dependent term in the Schrödinger equation.

We start with the following Schrödinger equation:

$$i \frac{d}{dt} |\Phi\rangle = (H_0 + f(t)H_1) |\Phi\rangle, \quad (58)$$

where  $H_0$  and  $H_1$  are time-independent, and rewrite it in the interaction picture (with respect to  $H_0$ ):

$$|\Phi\rangle = e^{-iH_0(t-t_0)} |\xi\rangle, \quad i \frac{d}{dt} |\xi\rangle = f(t) e^{iH_0(t-t_0)} H_1 e^{-iH_0(t-t_0)} |\xi\rangle. \quad (59)$$

We then proceed to consider

$$\begin{aligned} \frac{d}{dt} \left| |\xi(t)\rangle - |\xi(t_0)\rangle \right|^2 &= -\frac{d}{dt} (\langle \xi(t_0) | \xi(t) \rangle + c.c.) \\ &= -if(t) (\langle \xi(t_0) | e^{iH_0(t-t_0)} H_1 e^{-iH_0(t-t_0)} |\xi(t)\rangle - c.c.). \end{aligned} \quad (60)$$

Integrating this expression between  $t_0$  and  $t_0 + T$  and making use of standard inequalities

for absolute values and scalar products, we obtain:

$$\begin{aligned}
\left| |\xi(t_0 + T)\rangle - |\xi(t_0)\rangle \right|^2 &= -i \int_{t_0}^{t_0+T} dt f(t) (\langle e^{iH_0(t-t_0)} H_1 e^{-iH_0(t-t_0)} \xi(t_0) | \xi(t) \rangle - c.c.) \\
&\leq 2 \int_{t_0}^{t_0+T} dt |f(t)| \sqrt{(\langle e^{iH_0(t-t_0)} H_1 e^{-iH_0(t-t_0)} \xi(t_0) \rangle)^2} \sqrt{(|\xi(t)\rangle)^2} \\
&\leq 2 (\max_{[t_0, t_0+T]} |f(t)|) \int_0^T dt \sqrt{\langle \xi(t_0) | e^{iH_0 t} H_1^2 e^{-iH_0 t} | \xi(t_0) \rangle}.
\end{aligned} \tag{61}$$

Now, assume that  $f(t)$  approaches 0 at large times and consider a fixed  $|\xi(t_0)\rangle \equiv |\xi_0\rangle$  (so we consider the evolution with fixed duration  $T$  of the same initial state  $|\xi_0\rangle$  starting at different initial times  $t_0$ ). In this case, the first factor in the last line becomes arbitrarily small for large  $t_0$ , whereas the second factor does not depend on  $t_0$ . We then conclude that, for sufficiently late times, the finite time evolution of the state vector will be approximated arbitrarily well by  $|\xi(t)\rangle = \text{const}$ , i.e., by the evolution with  $f(t)$  set identically to 0.

It is then a simple corollary of the above bound that the time-dependent matrix theory dynamics becomes approximated arbitrarily well at late times by the flat space matrix theory, and, in particular, the supersymmetry is asymptotically restored (with all the usual consequences, such as protection of the flat directions of the commutator potential, and free graviton propagation).

Similar considerations can be given for the matrix string case, though they are more involved and rely on derivations in the style of quantum adiabatic theory. We refer the reader to the original publication [53].

## 4 Conclusions

We have reviewed some recent considerations of light-like singularities in string and matrix theories. In the presence of such singularities, these theories need to be supplemented with well-motivated prescriptions for singularity transition. A natural class of such prescriptions emerges from geometrical resolutions of the light-like singularities for which resolved space-times satisfy supergravity equations of motion. Important open questions include analysis of the singular limit for the full non-linear evolution of matrix theories.

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